Synthetic Projective Treatment of Cevian Nests and Graves Triangles

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1 Introduction

Several proofs of the cevian nest theorem (given below) are known, including one using ratios along sides and Ceva's theorem and another using Menelaus's theorem for quadrilaterals [4]. A synthetic proof using only Desargues's and Pappus's theorems has recently been published as well [1]. Here we give another synthetic projective geometry proof, one that uses conics. We also explore the relationship between cevian nests and Graves triangles, a cycle of three triangles each inscribed in the next. In particular, given $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, we give four different characterizations for the triangles $A_3B_3C_3$ inscribed in $A_2B_2C_2$ which complete a Graves cycle of triangles. The key is to simplify things by using the unique conic $\mathscr{C}$ which touches B_1C_1 at A_2, C_1A_1 at B_2, and A_1B_1 at C_2.

Throughout this note, we use the same notation used (among others) by Coxeter in *The Real Projective Plane* [6] and *Projective Geometry* [5]. Namely, points are denoted by capital letters, lines by lowercase letters. The line joining the two points A and B is denoted AB, and the intersection of the two lines a and b is denoted $a \cdot b$. If a line and a point are denoted by the same letter (lowercase and uppercase, respectively), then either they are related by the relevant polarity or they are the perspectrix and perspector, respectively, for the same pair of triangles. This will be clear from the context. The statement $H(AB, CD)$ means that C is the harmonic conjugate of D with respect to A and B. Whenever we refer to triangles, we assume they are nondegenerate (that is, the vertices are not collinear and the sides are not concurrent). Finally, we use the term "perspector" for the center of perspectivity (of two triangles) and "perspectrix" for the axis of perspectivity.

2 The Cevian Nest

Theorem 1. *Let A_1, B_1, C_1 be the vertices of a triangle; A_2, B_2, C_2 the vertices of a triangle inscribed in $A_1B_1C_1$ (so that A_2 is on B_1C_1, B_2 is on C_1A_1, and C_2 is on A_1B_1); and A_3, B_3, C_3 the vertices of a triangle inscribed in $A_2B_2C_2$, the points lying on the sides of the other in a similar fashion. Then if any two of the following three statements hold, so does the third:*

(1) A_2A_3, B_2B_3, C_2C_3 are concurrent at a point P.
(2) A_3A_1, B_3B_1, C_3C_1 are concurrent at a point Q.
(3) A_1A_2, B_1B_2, C_1C_2 are concurrent at a point R.

If any two (thus all three) of the above statements hold, the three triangles $A_iB_iC_i, i = 1, 2, 3$, are said to form a ***cevian nest****.*

Note that if $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, that is, the lines A_1A_2, B_1B_2, C_1C_2 concur, then since A_2, B_2, C_2 are the vertices of a triangle, none of these can be a vertex of $A_1B_1C_1$, and similarly for $A_3B_3C_3$. If, say, $A_2 = B_1$, then A_1A_2, B_1B_2, and C_1C_2 would have to concur on the side A_1B_1, which forces the points A_2, B_2, and C_2 to lie on a single line.

Proof. We first assume (3) and show that (1) holds if and only if (2) does. Let D be the harmonic conjugate of A_3 with respect to B_2C_2, E the harmonic conjugate of B_3 with respect to C_2A_2, and F the harmonic conjugate of C_3 with respect to A_2B_2. Furthermore, let $\mathscr{C}$ be the conic touching B_1C_1 at A_2, C_1A_1 at B_2, and A_1B_1 at C_2. This conic exists because any pair of Desargues triangles (specifically $A_1B_1C_1$ and $A_2B_2C_2$ here) are polar triangles under a certain polarity. (This is 5.71 of [6], but its proof in [6] does not rely on the order axioms, so it holds in any nontrivial projective space. The proof uses Hesse's theorem, but this is 7.61 in [5], so it does not rely on the order axioms either.) Under this polarity, the triangle $A_2B_2C_2$ would lie on the conic since each vertex lies on its own polar (e.g. A_2 on B_1C_1).

The key fact to note is that, with respect to the conic $\mathscr{C}$, D is the conjugate of A_3 along B_2C_2, E is the conjugate of B_3 along C_2A_2, and F is the

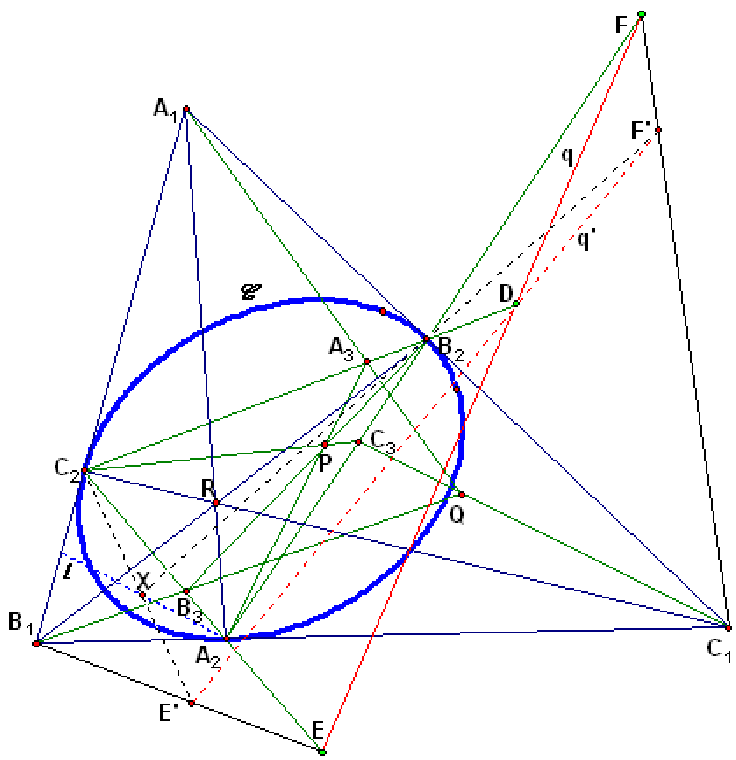

Figure 1: Cevian Nest

conjugate of C_3 along A_2B_2. This is because D is the harmonic conjugate of A_3 with respect to the two self-conjugate points B_2 and C_2 along the respective line (8.11 in [5]).

By definition of the conic, $b_2 \cdot c_2 = A_1$, so $B_2C_2 = a_1$. We have $A_3 = A_1Q \cdot B_2C_2$, so $a_3 = (a_1 \cdot q)(b_2 \cdot c_2) = (a_1 \cdot q)A_1$. In particular, a_3, a_1, and q are concurrent. But D is conjugate to A_3 on the line B_2C_2, so D lies on a_3. Since D also lies on $B_2C_2 = a_1$, D must be the point of intersection of a_3, a_1, and q. If we assume (2) then similarly, E is the point of intersection of b_3, b_1, and q, and F is the point of intersection of c_3, c_1, and q. The line through D, E, and F is then the polar q of Q because each of D, E, and F lies on q, and the point P is the trilinear pole of this line. Conversely, if we start with (1), we first find that the line through D, E, and F is the trilinear polar of P, then find its pole. Since $D = a_3 \cdot a_1$, $d = A_3A_1$, etc. so the fact that D, E, and F are collinear implies that A_3A_1, B_3B_1, and C_3C_1 are concurrent at a point Q.

Now we prove that (1) and (2) imply (3). Let D, E, and F be as above and let $\mathscr{C}$ be the conic touching C_1A_1 at B_2, A_1B_1 at C_2, and passing through A_2 (such a unique conic exists by 8.41 in [5]). We must show B_1C_1 is the polar a_2 of A_2, for then by Chasles's theorem (7.31 in [5]) the polar triangles $A_1B_1C_1 = a_2b_2c_2$ and $A_2B_2C_2$ would be Desargues triangles.

By the same argument given above, D is the point of intersection of a_3, a_1, and q. Now let q' be a variable line through D, and let $E' = q' \cdot b_3, F' = q' \cdot c_3$. Consider the locus of the intersection $X = E'C_2 \cdot F'B_2$ as q' varies. We have $E' \underset{\wedge}{\overset{D}{=}} F'$, so the relationship of pencils $E'C_2 \barwedge F'B_2$ holds. But the line B_2C_2 is fixed since D lies on B_2C_2, and a projectivity between two pencils is a perspectivity if the line joining the pencils is fixed (dual of 4.22 in [5]). Thus the locus of $E'C_2 \cdot F'B_2$ is a line.

We claim this line l intersects a_2 precisely in A_2. First suppose $l = a_2$. Then $c_2 \cdot a_2$ is on l. Since B_3 is on C_2A_2, the lines b_3, c_2, and a_2 are concurrent. Thus if $X = E'C_2 \cdot F'B_2 = c_2 \cdot a_2$ then the fact that E' is collinear with C_2 and $c_2 \cdot a_2$, yet lies on b_3, which goes through $c_2 \cdot a_2$ implies E' must be $c_2 \cdot a_2$, or else C_2 must be on b_3. The latter is impossible because C_2 is on c_2, so C_2 would be the intersection of b_3, c_2, and a_2, contradicting the fact that A_2 is the only point on the conic which lies on a_2. Also, F' must be $(c_2 \cdot a_2)B_2 \cdot c_3$. But D, E', and F' are collinear, i.e. D is collinear with $c_2 \cdot a_2$ and $(c_2 \cdot a_2)B_2 \cdot c_3$, so D lies on $(c_2 \cdot a_2)B_2$. Similarly, since $a_2 \cdot b_2$ is on l we see that D lies on $(a_2 \cdot b_2)C_2$. But D lies on B_2C_2, so this implies $B_2, C_2, c_2 \cdot a_2$, and $a_2 \cdot b_2$ are all collinear, i.e. $B_2C_2 = a_2$, again a contradiction since A_2 is the only point on the conic which lies on a_2.

Now we show A_2 lies on l. Hypothesis (1) implies the line through D, E, and F is the trilinear polar of R. If we let $q' = EF$ then $E' = E$ because E is the conjugate of B_3 on C_2A_2, so E lies on b_3. Similarly, $F' = F$. Then $X = E'C_2 \cdot F'B_2 = EC_2 \cdot FB_2 = A_2$.

Finally, by (2) and the definition of $\mathscr{C}$, $B_1 = B_3Q \cdot c_2$, so we have $b_1 = (b_3 \cdot q)C_2$ and, similarly, $c_1 = (c_3 \cdot q)B_2$. Hence, by taking $q' = q$ above, we see $b_1 \cdot c_1$ lies on l. But B_1, C_1, and A_2 are collinear, so b_1, c_1, and a_2 are concurrent. Therefore $b_1 \cdot c_1 = l \cdot a_2 = A_2$, by the above argument, or dually $B_1C_1 = a_2$, just as we needed. $\square$

Before moving on, we exhibit the definitions related to cevian nests in the Encyclopedia of Triangle Centers [2] in terms of our P, Q, and R as in theorem 1:

- P is the cevapoint of Q and R with respect to $A_2B_2C_2$.
- Q is the R-Ceva conjugate of P with respect to $A_2B_2C_2$.
- P is the crosspoint of R and Q with respect to $A_1B_1C_1$.
- Q is the P-cross conjugate of R with respect to $A_1B_1C_1$.

3 Graves Triangles and Perspectors Lying on Perspectrices

Recall that a cycle of Graves triangles is a series of three triangles $\Delta_1, \Delta_2, \Delta_3$ such that each is inscribed in the next: Δ_3 in Δ_2, Δ_2 in Δ_1, and Δ_1 in Δ_3. In this section we prove that in a cycle of Graves triangles, if one pair of triangles is a pair of Desargues triangles, then all are. In addition, if $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, we give two characterizations for all triangles $A_3B_3C_3$ that can be inscribed in $A_2B_2C_2$ to complete a cycle of Graves triangles with $A_1B_1C_1$ and $A_2B_2C_2$.

Exercise 6 of §4.4 in [5] states: "If one triangle is inscribed in another, any point on a side of the former can be used as a vertex of a third triangle which completes a cycle of Graves triangles." For completeness, we prove this here (see figure 2). As a matter of fact, we will see the same pattern of proof in the theorems that follow, so this is a good warm-up. Suppose $A_2B_2C_2$ is inscribed in $A_1B_1C_1$. For any point A_3 on B_2C_2, except B_2, C_2, and $B_2C_2 \cdot B_1C_1$, let $B_3 = C_1A_3 \cdot C_2A_2$ and $C_3 = A_3B_1 \cdot A_2B_2$. To prove $A_3B_3C_3$ completes the cycle of Graves triangles, we just have to show A_1 lies on B_3C_3, which the Pappus hexagon $C_1A_3B_1C_2A_2B_2$ does. By permuting the letters A, B, C, a similar proof shows the result if we had started with B_3 on C_2A_2 or C_3 on A_2B_2.

Now suppose the starting triangles in the exercise had been Desargues, too. Then we can still arrive at a Graves cycle of triangles by starting with a vertex on any side of $A_2B_2C_2$, but we would have more.

Theorem 2. *Let $A_iB_iC_i, i = 1, 2, 3,$ be a series of Graves triangles such that $A_1A_2, B_1B_2,$ and C_1C_2 are concurrent. Then the three triangles form a cevian nest, that is, $A_iA_j, B_iB_j,$ and C_iC_j are concurrent for all $i \neq j$. In this case, we propose to call this cycle of triangles a **Graves cevian nest**.*

Proof. If we look on the lines B_1C_1 and B_2C_2, the Pappus hexagon $A_2C_2C_1A_3B_1B_2$ shows that $A_2C_2 \cdot A_3B_1, C_2C_1 \cdot B_1B_2$, and $C_1A_3 \cdot B_2A_2$ are collinear. The first point is better called $C_2A_2 \cdot C_3A_3$; the second point is the perspector of $A_1B_1C_1$ and $A_2B_2C_2$; and the third point is better called $A_2B_2 \cdot A_3B_3$. Now if we look on the lines C_1A_1 and C_2A_2, the Pappus hexagon $B_2A_2A_1B_3C_1C_2$ shows that $B_2A_2 \cdot B_3C_1, A_2A_1 \cdot C_1C_2$, and $A_1B_3 \cdot C_2B_2$ are collinear. The first point is again $A_2B_2 \cdot A_3B_3$; the second point is again the perspector of $A_1B_1C_1$ and $A_2B_2C_2$; and the third point is now $B_2C_2 \cdot B_3C_3$. Thus we see that the points $A_2B_2 \cdot A_3B_3$, $C_2A_2 \cdot C_3A_3$, and $B_2C_2 \cdot B_3C_3$ are collinear with the perspector of $A_1B_1C_1$ and $A_2B_2C_2$. Since the intersections of the corresponding sides are collinear, the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are Desargues. By theorem 1, the triangles $A_1B_1C_1$ and $A_3B_3C_3$ are Desargues as well. □

Corollary 1. *If $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$ then any point on a side of $A_2B_2C_2$ can be used to create a third triangle $A_3B_3C_3$ which forms a Graves cevian nest with $A_1B_1C_1$ and $A_2B_2C_2$.*

Proof. This just combines exercise 6 of §4.4 in [5] (proved above) and theorem 2. □

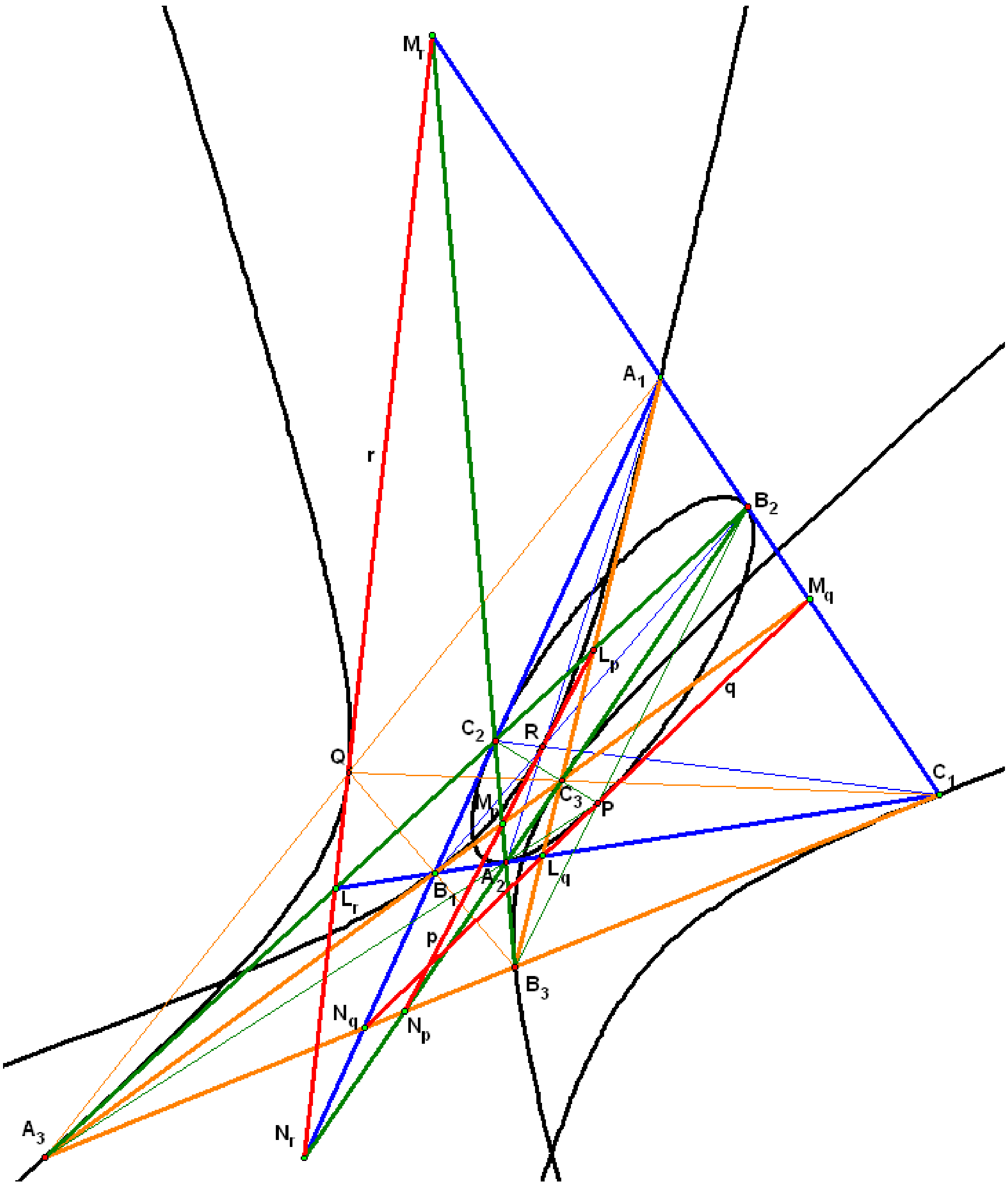

Figure 2: Graves Cevian Nest

Theorem 3. *Let $A_iB_iC_i, i = 1, 2, 3,$ be three triangles in a cevian nest, so that $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$ and $A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$. Then the following are equivalent (the indices are taken mod 3):*

(1) The triangles $A_iB_iC_i, i = 1, 2, 3,$ form a cycle of Graves triangles, that is, $A_1B_1C_1$ is also inscribed in $A_3B_3C_3$.
(2) The perspector of $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ lies on the perspectrix of $A_{i+1}B_{i+1}C_{i+1}$ and $A_{i-1}B_{i-1}C_{i-1}$, for all $i \in \{1, 2, 3\}$.
(3) The perspector of $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ lies on the perspectrix of $A_{i+1}B_{i+1}C_{i+1}$ and $A_{i-1}B_{i-1}C_{i-1}$, for some $i \in \{1, 2, 3\}$.

Proof. (1) $\Rightarrow$ (2): The proof of theorem 2 shows the result for $i = 1$. By cyclically permuting the indices, the same proof shows the result to be true for each $i \in \{1, 2, 3\}$.
(2) $\Rightarrow$ (3): This is trivial since (3) is a special case of (2).
(3) $\Rightarrow$ (1):

Suppose the perspector P of $A_2B_2C_2$ and $A_3B_3C_3$ lies on the perspectrix q of $A_3B_3C_3$ and $A_1B_1C_1$. Let the points on q be $L_q = B_3C_3 \cdot B_1C_1, M_q = C_3A_3 \cdot C_1A_1$, and $N_q = A_3B_3 \cdot A_1B_1$. The Pappus hexagon $B_2PC_2N_qA_3M_q$ shows $B_2P \cdot N_qA_3 = B_3, PC_2 \cdot A_3M_q = C_3$, and $C_2N_q \cdot M_qB_2 = A_1$ are collinear. Using the inherent symmetry, by cyclically permuting the letters A, B, C (and thus also L_q, M_q, N_q), we can get two other Pappus hexagons which show similarly that C_3, A_3, and B_1 are collinear, and that A_3, B_3, and C_1 are collinear.

Suppose the perspector Q of $A_3B_3C_3$ and $A_1B_1C_1$ lies on the perspectrix r of $A_1B_1C_1$ and $A_2B_2C_2$. Let the points on r be $L_r = B_1C_1 \cdot B_2C_2, M_r = C_1A_1 \cdot C_2A_2$, and $N_r = A_1B_1 \cdot A_2B_2$. The Pappus hexagon $B_1QC_1M_rA_2N_r$ shows $B_1Q \cdot A_2M_r = B_3, QC_1 \cdot A_2N_r = C_3$, and $C_1M_r \cdot N_rB_1 = A_1$ are collinear. By cyclically permuting the letters A, B, C, we can prove the other two collinearities.

Suppose the perspector R of $A_1B_1C_1$ and $A_2B_2C_2$ lies on the perspectrix p of $A_2B_2C_2$ and $A_3B_3C_3$. Let the points on p be $L_p = B_2C_2 \cdot B_3C_3, M_p = C_2A_2 \cdot C_3A_3$, and $N_p = A_2B_2 \cdot A_3B_3$. We cannot use Pappus hexagons alone here because we do not know, for example, that C_3 lies on B_1M_p, so we have to be a little tricky. Since $A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$, we have $H(B_3M_p, C_2A_2)$, and since $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, we have $H(A_1B_1, C_2N_r)$, where $N_r = A_1B_1 \cdot A_2B_2$ as before. Any two harmonic sets are related by a unique projectivity (4.21 in [5]), so $B_3M_pC_2A_2 \barwedge A_1B_1C_2N_r$. The intersection C_2 of these two ranges is fixed, so this projectivity is a perspectivity (4.22 in [5]): $B_3M_pC_2A_2 \overset{=}{\barwedge} A_1B_1C_2N_r$. This implies A_1B_3, B_1M_p, and $N_rA_2 = A_2B_2$ are concurrent. We will use this concurrency in a moment.

The Pappus hexagon $A_1RB_1M_pC_2L_p$ shows that $A_1R \cdot M_pC_2 = A_2$, $RB_1 \cdot C_2L_p = B_2$, and $B_1M_p \cdot A_1L_p$ are collinear. In other words, A_2B_2, B_1M, and A_1L_p are concurrent. But we just saw that A_2B_2 and B_1M_p are concurrent with A_1B_3, so either $A_1 = A_2B_2 \cdot B_1M_p$ is the point of concurrency, or A_1, B_3, and L_p are collinear. The former is impossible because A_1 cannot lie on A_2B_2 ($A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$). But B_3 and L_p are collinear with C_3, so either $L_p = B_3$ or A_1, B_3, and C_3 are collinear. The former is impossible because B_3 cannot lie on A_2B_2 ($A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$). Therefore, A_1 lies on B_3C_3. By cyclically permuting the letters A, B, C, we can similarly show B_1 lies on C_3A_3 and C_1 lies on A_3B_3. □

Corollary 2. *Suppose $A_2B_2C_2$ is the cevian triangle of $A_1B_1C_1$ with perspector R and $A_3B_3C_3$ is a triangle inscribed in $A_2B_2C_2$. The triangles $A_iB_iC_i, i = 1, 2, 3$, form a Graves cycle if and only if $A_2B_2C_2$ and $A_3B_3C_3$ are perspective from an axis that goes through R.*

Proof. If the triangles form a Graves cycle, theorem 2 says they form a cevian, so we can apply theorem 3. Conversely, if the perspectrix of $A_2B_2C_2$ and $A_3B_3C_3$ goes through R, then in particular the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are Desargues, so theorem 1 says the three triangles form a cevian nest and we can again apply theorem 3. □

Corollary 2 gives another characterization of the cevian triangles $A_3B_3C_3$ which complete a cycle of Graves triangles with $A_1B_1C_1$ and $A_2B_2C_2$ given $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$. This characterization shows that we can pick any line p through R (and not through A_2, B_2, or C_2) and get the triangle $A_3B_3C_3$ desired by finding the trilinear pole P of p with respect to $A_2B_2C_2$.

Before moving on, we feel we must note the duality of Figure 2 when the three conics are removed from it. There are 9 points (the vertices of the triangles), each of which lies on 5 lines, and 9 lines (the sides of the triangles), each of which passes through 5 points. There are 9 further points (the intersections of corresponding sides of the triangles), each of which lies on 3 lines, and 9 further lines (the joins of corresponding vertices of the triangles), each of which passes through 3 points. Finally, there are 3 further points (the perspectors), each of which lies on 4 lines, and 3 further lines (the perspectrices) each of which lies on 4 points.

4 Graves Triangles and Conics

Now we introduce the conic $\mathscr{C}$ from the proof of theorem 1 into the picture. First, in theorem 4, we show a natural example of a Graves cevian nest associated with $\mathscr{C}$ (see Figure 3). Then we give two theorems similar to theorem 3 and thus produce, given $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, another two characterizations of all triangles which can be inscribed in $A_2B_2C_2$ to complete the cycle of Graves triangles.

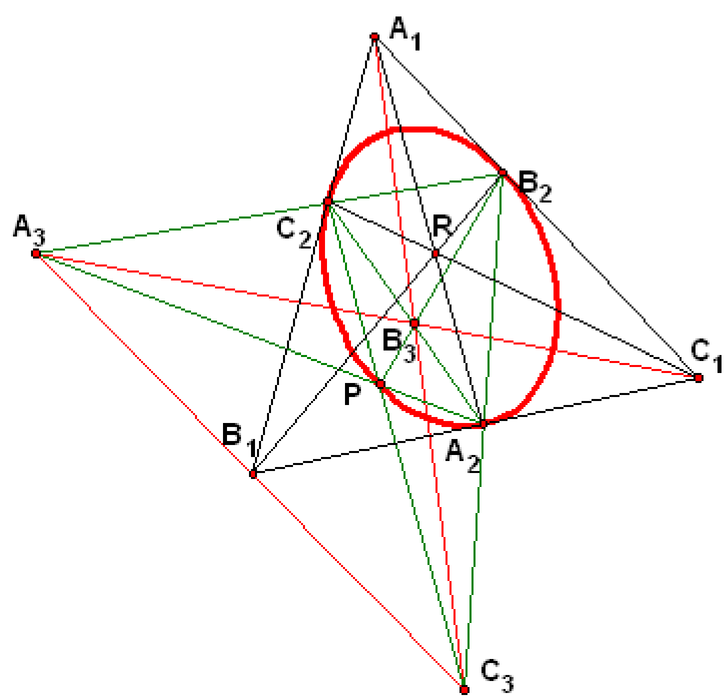

Figure 3: Graves Cevian Nest Associated with a Conic

Theorem 4. *Suppose $A_1B_1C_1$ is a tangential triangle to a conic $\mathscr{C}$ and the points of contact are $A_2, B_2,$ and C_2. For any point $P \in \mathscr{C} \smallsetminus \{A_2, B_2, C_2\}$, let $A_3B_3C_3$ be the diagonal triangle of the quadrangle $A_2B_2C_2P$. Then the triangles $A_iB_iC_i, i = 1, 2, 3$, form a Graves cevian nest.*

Proof. All we have to show is that $A_1B_1C_1$ is inscribed in $A_3B_3C_3$ because the rest follows from theorem 2. To show A_1 lies on B_3C_3, note that A_3 lies on $B_2C_2 = a_1$, so A_1 lies on $a_3 = B_3C_3$, and similarly for the other two sides. □

In the following theorem, we give a converse and thus more conditions that are equivalent to the formation of a Graves cevian nest from a cevian nest.

Theorem 5. *Let $A_iB_iC_i, i = 1, 2, 3$, be three triangles in a cevian nest, so that $A_2B_2C_2$ is inscribed in $A_1B_1C_1$ and $A_3B_3C_3$ is inscribed in $A_2B_2C_2$. Then the following are equivalent (the indices are taken mod 3):*

(1) The triangles $A_iB_iC_i, i = 1, 2, 3$, form a cycle of Graves triangles, that is, $A_1B_1C_1$ is also inscribed in $A_3B_3C_3$.
(2) The perspector of $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ lies on the conic $\mathscr{C}$ touching $A_{i-1}B_{i-1}C_{i-1}$ at $A_i, B_i,$ and C_i, for all $i \in \{1, 2, 3\}$.
(3) The perspector of $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ lies on the conic $\mathscr{C}$ touching $A_{i-1}B_{i-1}C_{i-1}$ at $A_i, B_i,$ and C_i, for some $i \in \{1, 2, 3\}$.

Proof. (1) $\Rightarrow$ (2): We prove the result for $i = 2$; the other two cases can be proven by cyclically permuting the indices $1, 2, 3$. Suppose $\mathscr{C}$ touches B_1C_1 at A_2, C_1A_1 at B_2, and A_1B_1 at C_2. We need to prove $P = A_2A_3 \cdot B_2B_3$ lies on $\mathscr{C}$. We do this by showing $A_3B_3C_3$ is a self-polar triangle. Since $A_3B_3C_3$ is the diagonal triangle of the quadrange $A_2B_2C_2P$, we know P is the harmonic conjugate of A_2 with respect to $PA_2 \cdot B_3C_3$ and A_3. If $A_3B_3C_3$ were self-polar, these last two points would be conjugate. Because conjugate points on a secant of a conic are harmonic conjugates with respect to the two self-conjugate points on the secant, P would then have to be a self-conjugate point and lie on $\mathscr{C}$.

We must show $B_3C_3 = a_3$. We know $A_3 = A_1Q \cdot B_2C_2$, so $a_3 = (a_1 \cdot q)(b_2 \cdot c_2) = (B_2C_2 \cdot q)A_1$. Because of (1) we know A_1 lies on B_3C_3, so we just have to show $B_2C_2 \cdot q$ also lies on B_3C_3. But, just as in the proof of theorem 1, q is

the trilinear polar of P, so $B_2C_2 \cdot q$ is the harmonic conjugate of $A_3P \cdot B_2C_2$ and thus lies on B_3C_3 because $A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$.
(2) $\Rightarrow$ (3): This is again trivial since (3) is a special case of (2).
(3) $\Rightarrow$ (1): If $i = 1$, the condition that a conic touching $A_3B_3C_3$ at A_1, B_1, and C_1 even exists immediately implies $A_1B_1C_1$ is inscribed in $A_3B_3C_3$, so there is nothing to prove.

If $i = 2$, then (1) immediately follows from theorem 4.

If $i = 3$, then $Q = A_1A_3 \cdot B_1B_3$ lies on the conic $\mathscr{C}$ touching $A_2B_2C_2$ at A_3, B_3, and C_3. This time, though $A_3B_3C_3Q$ is a quadrangle inscribed in $\mathscr{C}$, we do not know its diagonal triangle is $A_1B_1C_1$ yet. If we can prove this, then the fact that $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$ would imply $a_1b_1c_1 = A_1B_1C_1$ is a cevian triangle of $a_2b_2c_2 = A_3B_3C_3$, so the vertices of $A_1B_1C_1$ would lie on the sides of $A_3B_3C_3$.

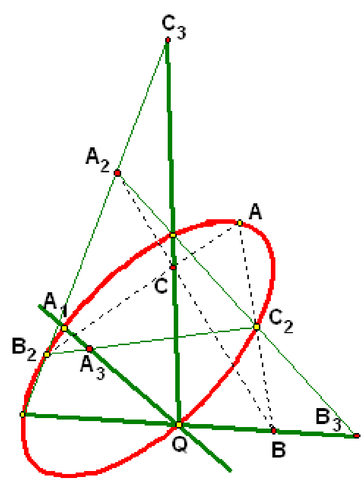

Figure 4: Anticevian Triangles of $A_2B_2C_2$ Lying on A_3Q, B_3Q, C_3Q

By theorem 4, $A_2B_2C_2$ is a cevian triangle of the diagonal triangle of $A_3B_3C_3Q$. We also know $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, whose vertices lie on the lines A_3Q, B_3Q, and C_3Q, so we just have to prove that there is at most one anticevian triangle of $A_2B_2C_2$ whose vertices lie on these lines. We will show there is at most one triangle ABC, with A on A_3Q, B on B_3Q, and C on C_3Q, such that $A_2B_2C_2$ is even inscribed in ABC (see Figure 4).

The strategy is to **not** restrict A to the line A_3Q and find the locus of all A in the plane such that $A_2B_2C_2$ is inscribed in triangle ABC for some

B on B_3Q and C on C_3Q, then intersect that locus with the line A_3Q. For any B on B_3Q, we can find C as $BA_2 \cdot C_3Q$ ($BA_2 \neq C_3Q$ because A_2 cannot lie on $C_3Q = C_1C_3$. If it did, then $C_1A_2 = B_1C_1$ would pass through Q, which would mean $C_1A_1 = C_3A_3$. But B_2 lies on C_1A_1, and $A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$, so B_2 cannot lie on C_3A_3.) Now we can find A as $CB_2 \cdot BC_2$. But this is exactly the Braikenridge-MacLaurin construction for a conic through five points (9.22 in [5]): ABC is a variable triangle with two of the vertices (namely B and C) lying on two fixed lines (B_3Q and C_3Q) and the three sides passing through three fixed points (the vertices of $A_2B_2C_2$, which are noncollinear by the assumption that $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$). The only thing to check is that B_3Q and C_3Q are not concurrent with B_2C_2, that is, Q does not lie on B_2C_2. If it did, then since A_3 also lies on B_2C_2, we would have $A_3A_1 = A_3Q = B_2C_2$. But $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, so A_1 cannot lie on B_2C_2. Thus, the locus of all possible A in the plane is a conic which passes through $B_2, C_2, Q, B_3Q \cdot A_2B_2$, and $C_3Q \cdot A_2C_2$. In particular, it intersects the line A_3Q in Q and at most one more point. Clearly, A cannot be Q itself - C_2 would again lie on A_3Q, which is impossible for the same reason A_2 cannot lie on C_3Q, as above. Thus, A can take at most one place on A_3Q, so there is at most one such triangle ABC in which $A_2B_2C_2$ can be inscribed. $\square$

Corollary 3. *Suppose $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$ and $A_3B_3C_3$ is a triangle inscribed in $A_2B_2C_2$. Let $\mathscr{C}$ be the conic touching $A_1B_1C_1$ at A_2, B_2, and C_2. The triangles $A_iB_iC_i, i = 1, 2, 3$, form a Graves cycle if and only if $A_2B_2C_2$ and $A_3B_3C_3$ are perspective from a center that lies on $\mathscr{C}$.*

Before moving on to the final characterization, we need another interesting lemma.

Lemma 1. *(1) If, for some polarity, a triangle $A_2B_2C_2$ is the cevian triangle of both the triangles $A_1B_1C_1$ and $a_1b_1c_1$, then $A_1B_1C_1$ is self-polar.*
(2) If, for some polarity, a triangle $A_2B_2C_2$ is the anticevian triangle of both the triangles $A_1B_1C_1$ and $a_1b_1c_1$, then $A_1B_1C_1$ is self-polar.

Proof. (1) Either (i) $a_1 = B_1C_1$ or (ii) $A_2 = a_1 \cdot B_1C_1$, since A_2 lies on a_1 and B_1C_1, and similarly for B_2 and C_2. It is impossible that (ii) holds for each of A_2, B_2, and C_2 since these are the intersections of the corresponding sides of $A_1B_1C_1$ and $a_1b_1c_1$, which, by Chasles's theorem, would be collinear. We can assume without loss of generality that $a_1 = B_1C_1$ since otherwise we

can rename the vertices. Then $A_1 = b_1 \cdot c_1$, so A_1 lies on b_1 and c_1. Now if $B_2 = b_1 \cdot C_1A_1$ then, since A_1 lies on b_1 and C_1A_1, we have $A_1 = B_2$, which is impossible since $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$. Thus $b_1 = C_1A_1$ and, similarly, $c_1 = A_1B_1$.
(2) Suppose $A_1B_1C_1$ is not self-polar. Both A_1 and $b_1 \cdot c_1$ lie on B_2C_2, and similarly for the other two sides of $A_2B_2C_2$. By Chasles's theorem $A_1B_1C_1$ and $a_1b_1c_1$ are Desargues, so $A_1(b_1 \cdot c_1) = B_2C_2$, $B_1(c_1 \cdot a_1) = C_2A_2$, and $C_1(a_1 \cdot b_1) = A_2B_2$ are concurrent, which contradicts the fact that $A_2B_2C_2$ is a triangle. □

It is curious to note that the proof of (1) requires that the vertices of $A_2B_2C_2$ are not collinear, while the proof of (2) requires that the sides are not concurrent.

The final theorem of this section is the basis for the last characterization of the triangles $A_3B_3C_3$ that complete a Graves cevian nest.

Theorem 6. *Let $A_i, B_i, C_i, i = 1, 2, 3$, be the vertices of a cevian nest, so that $A_2B_2C_2$ is inscribed in $A_1B_1C_1$ and $A_3B_3C_3$ is inscribed in $A_2B_2C_2$. Then the following are equivalent (the indices are taken mod 3):*

(1) The triangles $A_iB_iC_i, i = 1, 2, 3$, form a cycle of Graves triangles, that is, $A_1B_1C_1$ is also inscribed in $A_3B_3C_3$.
(2) The perspectrix of $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ is tangent to the conic $\mathscr{C}$ touching $A_{i+1}B_{i+1}C_{i+1}$ at A_{i-1}, B_{i-1}, and C_{i-1}, for all $i \in \{1, 2, 3\}$.
(3) The perspectrix of $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ is tangent to the conic $\mathscr{C}$ touching $A_{i+1}B_{i+1}C_{i+1}$ at A_{i-1}, B_{i-1}, and C_{i-1}, for some $i \in \{1, 2, 3\}$.

Proof. (1) ⇒ (2): We prove the result for $i = 3$; the other two cases can be proven by cyclically permuting the indices $1, 2, 3$. Suppose $\mathscr{C}$ touches B_1C_1 at A_2, C_1A_1 at B_2, and A_1B_1 at C_2. By the proof of theorem 5, the triangle $A_3B_3C_3$ is self-polar. Now the perspectrix of $A_3B_3C_3$ and $A_1B_1C_1$ is $(B_3C_3 \cdot B_1C_1)(C_3A_3 \cdot C_1A_1)$, so its polar is $A_3A_2 \cdot B_3B_2$, the perspector P of $A_2B_2C_2$ and $A_3B_3C_3$. Since the perspectrix is tangent to the conic, its polar lies on itself, and by theorem 5, P lies on $\mathscr{C}$. Thus, the perspectrix touches $\mathscr{C}$ at P.
(2) ⇒ (3): This is again trivial since (3) is a special case of (2).
(3) ⇒ (1): If $i = 1$, $\mathscr{C}$ touches $A_2B_2C_2$ at A_3, B_3, and C_3. The fact that $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$ implies $a_1b_1c_1$ is a cevian triangle of

$a_2b_2c_2 = A_3B_3C_3$. Since $A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$, the triangles $A_2B_2C_2, A_3B_3C_3$, and $a_1b_1c_1$ form a cevian nest. The pole of the perspectrix of $A_1B_1C_1$ and $A_2B_2C_2$ is the perspector of $a_1b_1c_1$ and $A_3B_3C_3$, and it lies on $\mathscr{C}$ because the perspectrix is self-polar. We can now apply theorem 5 to the cevian nest just mentioned to see that $A_2B_2C_2$ is a cevian triangle of $a_1b_1c_1$. But $A_2B_2C_2$ is also a cevian triangle of $A_1B_1C_1$, so by part (1) of lemma 1 $A_1B_1C_1$ is self-polar, which means $A_1B_1C_1 = a_1b_1c_1$ is a cevian triangle of $A_3B_3C_3$.

If $i = 2$, the condition that a conic touching $A_3B_3C_3$ at A_1, B_1, and C_1 even exists immediately implies $A_1B_1C_1$ is inscribed in $A_3B_3C_3$, so there is nothing to prove.

If $i = 3$, $\mathscr{C}$ touches $A_1B_1C_1$ at A_2, B_2, and C_2. Because $A_3B_3C_3$ is a cevian triangle of $A_2B_2C_2$, $a_2b_2c_2 = A_1B_1C_1$ is a cevian triangle of $a_3b_3c_3$, and this time the triangles $a_3b_3c_3, A_1B_1C_1$, and $A_2B_2C_2$ form the cevian nest we are interested in. We can again apply theorem 5 to this cevian nest to see that $a_3b_3c_3$ is a cevian triangle of $A_2B_2C_2$. But $A_3B_3C_3$ is also a cevian triangle of $A_2B_2C_2$, so by part (2) of lemma 1 $A_3B_3C_3$ is self-polar, which means $A_1B_1C_1$ is a cevian triangle of $a_3b_3c_3 = A_3B_3C_3$. □

Here at last is our final characterization of the $A_3B_3C_3$ triangles that complete a Graves cycle.

Corollary 4. *Suppose $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$ and $A_3B_3C_3$ is a triangle inscribed in $A_2B_2C_2$. Let $\mathscr{C}$ be the conic touching $A_1B_1C_1$ at A_2, B_2, and C_2. The triangles $A_iB_iC_i, i = 1, 2, 3$, form a Graves cycle if and only if $A_3B_3C_3$ and $A_1B_1C_1$ are perspective from an axis that is tangent to $\mathscr{C}$.*

5 Perspectrix $\leftrightarrow$ Conic Map

Now consider the perspectrix of $A_1B_1C_1$ and $A_2B_2C_2$ and the conic $\mathscr{C}$ touching $A_1B_1C_1$ at $A_2B_2C_2$, as in Figure 5. This perspectrix is the trilinear polar of the perspector R of $A_1B_1C_1$ and $A_2B_2C_2$ with respect to $A_2B_2C_2$ and also the polar r of R with respect to $\mathscr{C}$(we have seen this multiple times, especially in the proof of theorem 1). Combinining theorems 3 and 5, we see that the perspector Q of $A_1B_1C_1$ and $A_3B_3C_3$ lies on r if and only if the perspector P of $A_2B_2C_2$ and $A_3B_3C_3$ lies on the conic $\mathscr{C}$.

If we fix $A_1B_1C_1$ and P, the above argument gives us a bijective map (with an obvious inverse) from r to $\mathscr{C}$ as follows. If Q is any point on r, find $A_3B_3C_3$ by joining Q to the vertices of $A_1B_1C_1$ and finding the corresponding intersections with the sides of $A_2B_2C_2$. Then by theorem 1, $A_2B_2C_2$ and $A_3B_3C_3$ are perspective from a point P, and by theorems 3 and 5, P lies on the conic. In other words, P is the cevapoint of Q and R with respect to $A_2B_2C_2$. For the inverse map, if P lies on $\mathscr{C}$, let $A_3B_3C_3$ be the cevian triangle of P with respect to $A_2B_2C_2$. Then the perspector Q of $A_1B_1C_1$ and $A_3B_3C_3$ lies on r. In other words, Q is the R-Ceva conjugate of P with respect to $A_2B_2C_2$. Since the inverse of the map exists, it is bijective.

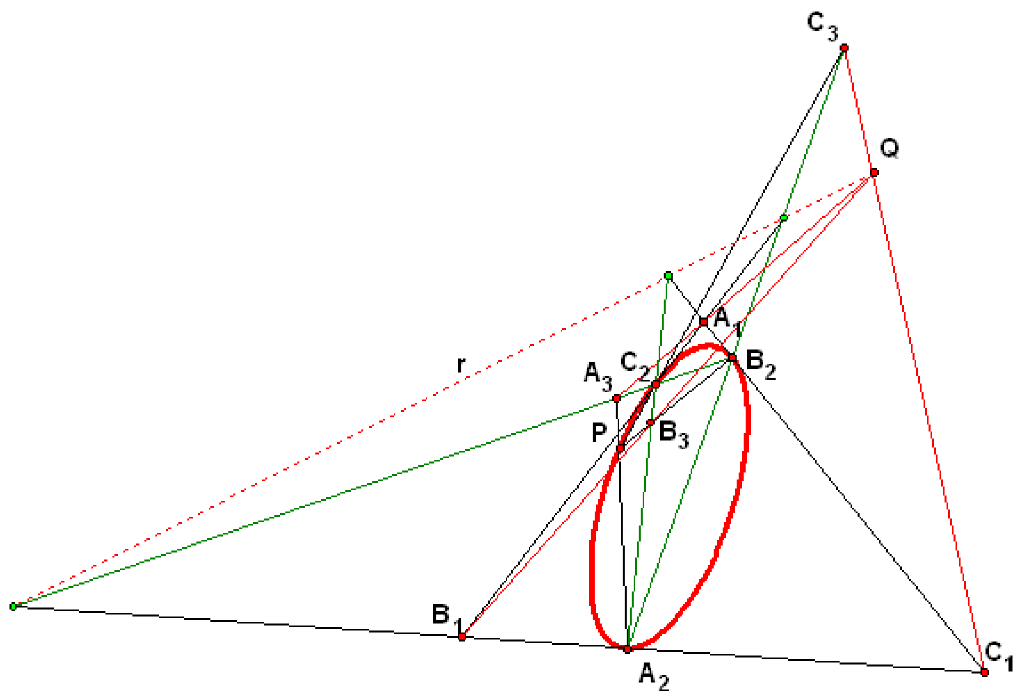

Figure 5: Perspectrix ↔ Conic Map

A somewhat different construction for the inverse of this map was given by Francois Rideau in [3] as follows (in our notation). Given a point P on $\mathscr{C}$, find its tangent (or polar) p. Then Q is the trilinear pole of p with respect to $A_1B_1C_1$ (see Figure 6 below). To prove this map is the same, we first note that in our description we connect A_1 to A_3 and similarly for the other vertices, then take the intersection of the resulting concurrent lines, and in his construction we connect A_1 to the harmonic conjugate of $p \cdot B_1C_1$ with respect to B_1 and C_1 instead, so we just have to show that these two lines are the same. That is, we have to show A_3 lies on the harmonic conjugate of $A_1(p \cdot B_1C_1)$ with respect to A_1B_2 and A_1C_2. This is because the harmonic conjugate of $A_3 = A_2P \cdot B_2C_2$ with respect to B_2 and C_2 is the conjugate point $(a_2 \cdot p)(b_2 \cdot c_2) \cdot B_2C_2 = (a_2 \cdot p)A_1 \cdot B_2C_2$.

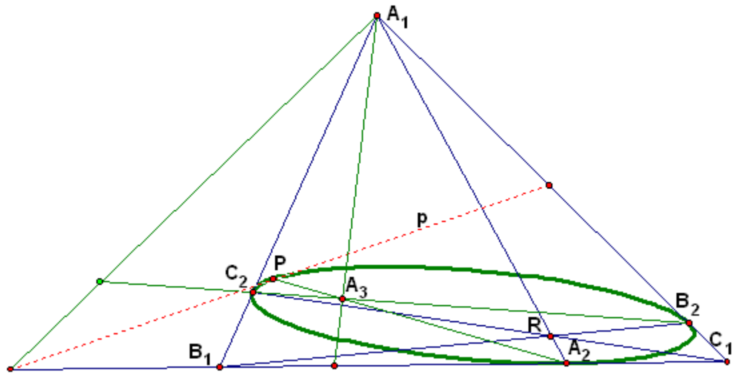

Figure 6: Rideau's Construction

The following is in response to the two questions asked in [3] then. If we know a triangle $A_1B_1C_1$ circumscribes a conic $\mathscr{C}$ and the points of contact A_2, B_2, and C_2, we have the following two projective constructions:

(1) Given P on $\mathscr{C}$, to construct the tangent to P, find the trilinear polar of $A_1(A_2P \cdot B_2C_2) \cdot B_1(B_2P \cdot C_2A_2)$ with respect to $A_1B_1C_1$.

(2) Given a tangent line p to $\mathscr{C}$, to find the point P of contact, first find the trilinear pole Q of p with respect to $A_1B_1C_1$. Then $P = A_2(A_1Q \cdot B_2C_2) \cdot B_2(B_1Q \cdot C_2A_2)$.

6 Two Cevian Triangles

This section deals with the situation when there are two cevian triangles DEF and $D'E'F'$ of ABC and the conic $\mathscr{C}$ that goes through all six vertices. Among other results, we prove there is exactly one triangle XYZ which simultaneously completes the two Graves cycles with ABC and DEF and with ABC and $D'E'F'$.

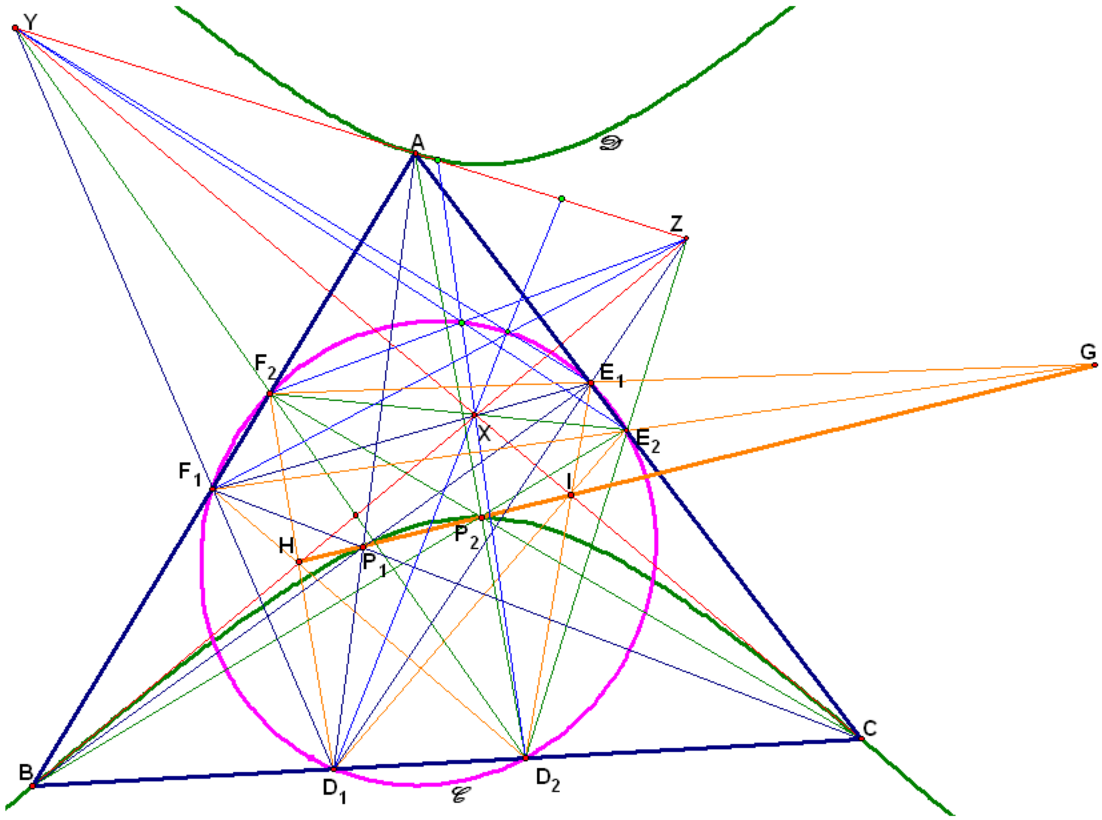

Figure 7: Two Cevian Triangles

Theorem 7. *Suppose DEF and $D'E'F'$ are two different cevian triangles of ABC with respective perspectors P and P'. Defining*

$$X = EF \cdot E'F', Y = FD \cdot F'D', Z = DE \cdot D'E',$$

XYZ is the unique triangle which simultaneously completes a Graves cycle with ABC and DEF and another Graves cycle with ABC and $D'E'F'$.

Proof. To show uniqueness, first note that by corollary 2 the perspector of ABC and any triangle $X_1Y_1Z_1$ completing a graves cycle with DEF must lie on the perspectrix of ABC and DEF. Similarly, this perspector must also lie on the perspectrix of ABC and $D'E'F'$. Since the two cevian triangles are not the same, the perspectrices are not the same, so they intersect in one point O. This point must be the perspector of ABC and $X_1Y_1Z_1$. In other words, $X_1Y_1Z_1$ must be the anticevian triangle of ABC with respect to O.

Since DEF is inscribed in ABC and XYZ is inscribed in both DEF and $D'E'F'$, we just have to show ABC is inscribed in XYZ. The following idea is due to Patrick Morton[1]. Let $\mathscr{D}$ be the conic through A, B, C, P, P'. Then DEF and $D'E'F'$, being diagonal triangles of quadrangles inscribed in $\mathscr{D}$, are self-polar with respect to $\mathscr{D}$. As $A = EE' \cdot FF', a = (e \cdot e')(f \cdot f') = (FD \cdot F'D')(DE \cdot D'E') = YZ$. Since A lies on $\mathscr{D}$, it lies on its polar $a = YZ$ as well. Similarly, B lies on $b = ZX$ and C on $c = XY$. □

Theorem 8. *In the situation of theorem 7, the traces DEF and $D'E'F'$ lie on a unique conic $\mathscr{C}$. Furthermore,*

1) *The triples of triangles XYZ, ABC, DEF and $XYZ, ABC, D'E'F'$ form Graves cevian nests.*

2) *The triangle XYZ is self-polar with respect to this conic.*

3) *The trilinear polars of P and P' with respect to ABC and the polars p and p' of P and P' with respect to $\mathscr{C}$ are concurrent at the perspector O of ABC and XYZ.*

4) *The perspectors of XYZ with DEF and $D'E'F'$, respectively, lie on $\mathscr{C}$.*

5) *The conic $\mathscr{D}$ above is the locus of all points P_1 whose cevian triangles on ABC complete the Graves cycle with XYZ and ABC.*

Proof. Let $G = EF'.E'F, H = FD'.F'D$, and $I = DE'.D'E$. The Pappus hexagon $BEF'CFE'$ shows P, G, P' are collinear. Similarly (by permuting the letters in A, B, C and D, E, F) we see that P, H, P' are collinear, and P, I, P' are collinear. Thus, the points G, H, I, P, P' are all collinear. By the converse of Pascal's theorem (essentially the Braikenridge-Maclaurin

[1]Private communication.

construction 9.22 of [5]) the hexagon $DE'FD'EF'$ shows that the points D, E, F, D', E', F' all lie on a conic. There are at least five distinct points among these (five if PP' goes through A, B, or C), and there is a unique conic through five points.

Now, 1) can be proven by theorem 2. Or, we could use Chasles's theorem on triangles ABC and XYZ with respect to the conic $\mathscr{D}$, since they are polar triangles by the proof of theorem 7, and then the cevian nest theorem (theorem 1). The Pascal hexagon $DFE'D'F'E$ shows Y, G, Z are collinear. Then since AGX is the diagonal triangle of $EE'FF'$, the polar of X is $AG = YZ$, and similarly for the other sides. Thus XYZ is self-polar. It was shown in the proof of theorem 7 that the trilinear polars of P and P' pass through O. The quadrangles $DD'EE'$ and $FF'DD'$ with diagonal triangles BHY and CIZ show that the polar of $O = BY \cdot CZ$ is $o = HI = PP'$, so $O = p \cdot p'$. If Q is the perspector of DEF and XYZ, D and Q are harmonic conjugates with respect to X and $DX \cdot YZ$ because XYZ is the diagonal triangle of $DEFQ$. But X is conjugate to $DX \cdot YZ$ since XYZ is self-polar, so Q must be the other intersection of DX with the conic, and similarly for the perspector of $D'E'F'$ and XYZ. The last remark is just an application of theorem 5. □

7 Conclusion

Given $A_2B_2C_2$ is a cevian triangle of $A_1B_1C_1$, we now have the following four different characterizations for the triangles $A_3B_3C_3$ which can be inscribed in $A_2B_2C_2$ to complete a Graves cycle. A vertex of one such triangle may lie anywhere on a given side of $A_2B_2C_2$ (except the vertices of $A_2B_2C_2$ and the intersections of corresponding sides of $A_1B_1C_1$ and $A_2B_2C_2$), and it is easy to derive the triangle given one vertex. The triangles $A_2B_2C_2$ and $A_3B_3C_3$ must be perspective, and the perspectrix may be any line through R but not through A_2, B_2, or C_2. The triangles $A_2B_2C_2$ and $A_3B_3C_3$ must be perspective, and the perspector may be any point other than A_2, B_2, or C_2 on the conic $\mathscr{C}$ touching $A_1B_1C_1$ at A_2, B_2, and C_2. Finally, the triangles $A_2B_2C_2$ and $A_3B_3C_3$ must be perspective, and the perspectrix may be any line tangent to this conic $\mathscr{C}$ other than the tangents at A_2, B_2, and C_2.

In particular, in a Graves cevian nest consisting of triangles Δ_1, Δ_2, and Δ_3, where Δ_{i+1} is a cevian triangle of Δ_i, if $\mathscr{C}$ is the conic touching Δ_i at the vertices of Δ_{i+1}, then the perspectrix of Δ_{i-1} and Δ_i touches $\mathscr{C}$ at the perspector of Δ_{i+1} and Δ_{i-1}, for each $i \in \{1, 2, 3\}$, where the indices are taken mod 3.

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